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A Branch-and-Bound Approach for Solving a Class of Generalized Semi-infinite Programming Problems

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Abstract. A nonconvex generalized semi-infinite programming problem is considered, involving parametric max-functions in both the objective and the constraints. For a fixed vector of parameters, the values of these parametric max-functions are given as optimal values of convex quadratic programming problems. Assuming that for each parameter the parametric quadratic problems satisfy the strong duality relation, conditions are described ensuring the uniform boundedness of the optimal sets of the dual problems w.r.t. the parameter. Finally a branch-and-bound approach is suggested transforming the problem of finding an approximate global minimum of the original nonconvex optimization problem into the solution of a finite number of convex problems.

Key words: Branch-and-bound algorithm, Nonconvex optimization, Nondifferentiable optimization, Quadratic programming problem, Semi-infinite optimization

1. Introduction

For an introduction to semi-infite programming problems we refer to the extensive survey in Hettich and Kortanek (1993). Concerning the so called *generalized semi-infinite programming* (GSIP), that is a problem where the index set of the constraints depends on the decision variables of the problem, in the recent past there has been a growing number of papers. In particular, we refer to Hettich et al. (1995), Weber (1996), and Jongen et al. (1998) for the generic structure of the solution set of GSIP and to Hettich and Still (1995) and Kaplan and Tichatschke (1996) for optimality conditions and special properties of GSIP. In Levitin and Tichatschke (1998) a smooting procedure for generalized max-functions is developed, which can be used in the framework of GSIP to work with the differential calculus for these non-smooth functions.

However, there are only a few papers dealing with some basic ideas for numerical methods for GSIP (see Graettinger and Krogh, 1988; Hettich and Still, 1991; Kaplan and Tichatschke, 1997). In the latter paper a special class of ill-posed GSIP, arising in robotics or time minimal control problems, is treated by means of a proximal point technique.

Here the following optimization problem is considered:

$$c_0(x) + m_0(x) \to \min, \quad x \in Q, \tag{1.1}$$

with

$$Q = \{x \in \mathcal{X} : \; c_i(x) + m_i(x) \leqslant 0 \; (i \in I)\}, \tag{1.2}$$

and, for $i \in I' := \{0\} \cup I$,

$$m_i(x) := \max\left\{\tfrac{1}{2}\langle G_i z_i, z_i\rangle + \langle \ell_i + H_i x, z_i\rangle : z_i \in D_i(x)\right\}, \tag{1.3}$$

$$\begin{aligned} D_i(x) := \{z_i \in \mathbf{Z}_i : &\langle p_{is}, z_i\rangle + q_{is}(x) \leqslant 0 \;\; (s \in S_{i1}), \\ &\langle p_{is}, z_i\rangle + q_{is}(x) = 0 \;\; (s \in S_{i2})\}. \end{aligned} \tag{1.4}$$

In this description we suppose:

$\mathcal{X}$ is a convex closed set in the Euclidean space $\mathbf{X}$;

I is a finite set of indices;

for each $i \in I'$ the functions c_i are continuous in $\mathbf{X}$ and convex on $\mathcal{X}$;

$G_i : \mathbf{Z}_i \to \mathbf{Z}_i$ are symmetric, negative semi-definite linear operators in the Euclidean spaces $\mathbf{Z}_i$;

$\ell \in \mathbf{Z}_i$ and $p_{is} \in \mathbf{Z}_i$ $(s \in S_i := S_{i1} \cup S_{i2})$ are vectors;

$H_i : \mathbf{X} \to \mathbf{Z}_i$ are arbitrary linear operators;

S_{i1} and S_{i2} are finite sets of indices;

the functions $-q_{is}$ $(s \in S_{i1})$ are convex on $\mathcal{X}$ and continuous on $\mathbf{X}$, and q_{is} $(s \in S_{i2})$ are affine on $\mathbf{X}$.

First, we assume that for each $x \in \mathbf{X}$ the set $D_i(x) \neq \emptyset$ for arbitrary $i \in I'$. Below, some conditions are given ensuring this property. Obviously, if in Problem (1.1), (1.2) $Q \neq \emptyset$ and there exists a point $x \in Q$ for which $c_0(x) + m_0(x) < +\infty$, then $m_i(x) < +\infty \; \forall \, i \in I'$.

In system analysis, in particular, in mathematical economics (Germeyer, 1976) and robotics (Graettinger and Krogh, 1988), often optimization models are considered with entries, depending on a vector of parameters z_i $(i \in I')$. For example, an objective function $\mathcal{J}$ and functions f_i, describing inequality constraints, may have the form

$$\mathcal{J}(x) := c_0(x) + h_0(x, z_0), \qquad f_i(x) := c_i(x) + h_i(x, z_i) \; (i \in I).$$

Each of the parameters z_i belongs to a corresponding set D_i, which may depend on the sought solution x of the initial problem, i.e. $D_i = D_i(x)$, where $D_i(x)$ is, for instance, given as in (1.4).

Usually it is unknown which vector $z_i \in D_i(x)$ has to be chosen, i.e., we deal with an optimization model under uncertainty. If there is no information about such a possible choice, it is natural to use the 'principle of guaranteed results' (cf. Germeyer, 1976a, b) or other conservative strategies (cf. Tichatschke et al., 1989), ensuring the validity of the model also in the worst situation. That means a solution $x \in \mathcal{X}$ is sought which satisfies the inequality $c_i(x) + h_i(x, z_i) \leqslant 0$ $(i \in I)$ for arbitrary $z_i \in D_i(x)$ and minimizes the objective function under the worst $z_0 \in D_0(x)$. In this case we obtain the following problem:

$$c_0(x) + \max\{h_0(x, z_0) : z_0 \in D_0(x)\} \to \min$$

$$\text{s.t.} \quad c_i(x) + \max\{h_i(x, z_i) : z_i \in D_i(x)\} \leqslant 0 \ (i \in I), x \in \mathcal{X}.$$

This problem is equivalent to Problem (1.1), (1.2) if we suppose that for all $i \in I'$ the functions h_i are of the form

$$h_i(x, z_i) := \tfrac{1}{2}\langle G_i z_i, z_i\rangle + \langle \ell_i + H_i x, z_i\rangle,$$

where the sets $D_i(x)$ are given via (1.4).

In this framework, for each $i \in I'$, a family of parametric, convex, quadratic programming problems (PQP) with parameter $x \in \mathcal{X}$ can be considered:

$$\tfrac{1}{2}\langle G_i z_i, z_i\rangle + \langle \ell_i + H_i x, z_i\rangle \to \max, \quad z_i \in D_i(x). \tag{1.5}$$

We call the function $m_i(x)$, giving via (1.3) for fixed $x \in \mathcal{X}$ the optimal value of Problem (1.5), *generalized max-function of type PQP*. From the theory of convex programming it follows that, in case $D_i(x) \neq \emptyset$ and $m_i(x) > -\infty$, the duality relation is always true. In general, due to the dependence of D_i on x, the optimal value function $m_i(x)$ of (1.5) is nonconvex. Hence, (1.1)-(1.4) is a nonconvex optimization problem and we call it *generalized semi-infinite programming problem of the type PQP*, because it can be rewritten as a problem in the space $\mathbf{X} \times \mathbb{R}^1$:

$$\begin{aligned}
&v \to \min \\
&x \in \mathcal{X}, v \in \mathbb{R}^1, \\
&c_0(x) + \tfrac{1}{2}\langle G_0 z_0, z_0\rangle + \langle \ell_0 + H_0 x, z_0\rangle - v \leqslant 0 \quad \forall z_0 \in D_0(x), \\
&c_i(x) + \tfrac{1}{2}\langle G_i z_i, z_i\rangle + \langle \ell_i + H_i x, z_i\rangle \leqslant 0 \qquad \forall z_i \in D_i(x) \ (i \in I).
\end{aligned} \tag{1.6}$$

The main idea of the paper is that under suitable assumptions, and by using the dual problems of the arising quadratic subproblems (1.5), a reduced nonconvex problem with a finite number of variables and constraints can be obtained (cf. (4.1)–(4.5)). In the latter problem, the convex envelopes of the nonconvex functions in the objective and constraints can be easily computed. A branch-and-bound approach is proposed for solving the latter problem. As a consequence, an approximate solution of the original problem can be obtained by solving finitely many convex problems.

We do not strive to describe in detail the resulting branch-and-bound algorithm for the transformed problems, because it does not lead to fundamental new theoretical or practical contributions. Rather, the basic message in this paper consists in the connection between some special generalized semi-infinite programming problems and the possibility of their reduction to a sequence of convex finite dimensional problems. To what extent this approach works efficiently in practice, is still an open question and needs computational experiences.

In Section 2 basic properties of Problem (1.5) and assumptions are given under which Problem (1.1)–(1.4) is investigated. An explicit form of the dual function for Problem (1.5) is described in Section 3 and some constants are calculated, ensuring that for all $i \in I'$ and $x \in Q$ the optimal sets of the dual problems to Problem (1.5) are contained in certain parallelepipeds. In Section 4 the equivalence between Problem (1.1)–(1.4) and a reduced problem (4.1)–(4.5) is proved. Finally, a branch-and-bound technique is sketched briefly to find an approximate solution of Problem (4.1)–(4.5) by means of the solutions of a finite number of convex problems (see (4.9)).

2. Basic notations and assumptions

Let $y_i = \{y_{is}\}_{s \in S_i} \in \mathbf{Y}_i = \mathbb{R}^{|S_i|}$ be the Lagrange multiplier vector for Problem (1.5) and

$$\mathcal{Y}_i = \mathbb{R}_+^{|S_{i1}|} \times \mathbb{R}^{|S_{i2}|}. \tag{2.1}$$

Denote

$$\Pi_i(x, z_i, y_i) := \tfrac{1}{2}\langle G_i z_i, z_i\rangle + \left\langle \ell_i + H_i x - \sum_{s \in S_i} y_{is} p_{is}, z_i \right\rangle - \sum_{s \in S_i} y_{is} q_{is}(x), \tag{2.2}$$

$$\varphi_i(x, y_i) := \sup\{\Pi_i(x, z_i, y_i) : z_i \in \mathbf{Z}_i\}, \tag{2.3}$$

the Lagrange function and dual function for Problem (1.5), respectively. It is obvious that

$$\varphi_i(x, y_i) := \pi_i(x, y_i) - \sum_{s \in S_i} y_{is} q_{is}(x), \tag{2.4}$$

with

$$\pi_i(x, y_i) := \sup\left\{ \tfrac{1}{2}\langle G_i z_i, z_i\rangle + \langle \ell_i + H_i x - \sum_{s \in S_i} y_{is} p_{is}, z_i\rangle : z_i \in \mathbf{Z}_i \right\}. \tag{2.5}$$

The function $\pi_i(x, y)$ is convex w.r.t. the pair $\{x, y_i\}$, because it is the supremum w.r.t. $z_i \in \mathbf{Z}_i$ of a parametric affine function in $\{x, y_i\}$. From (2.4) it follows that for each $i \in I'$ the dual to Problem (1.5) has the form

$$\pi_i(x, y_i) - \sum_{s \in S_i} y_{is} q_{is}(x) \to \min, \ \ y_i \in \mathcal{Y}_i. \tag{2.6}$$

Let

$$\mathcal{Y}_i^*(x) := Argmin \left\{ \pi_i(x, y_i) - \sum_{s \in S_i} y_{is} q_{is}(x) : y_i \in \mathcal{Y}_i \right\} \tag{2.7}$$

be the optimal set of Problem (2.6).

Neglecting the fact that it was necessary to describe some of the assumptions in the introduction in order to understand Problem (1.1)–(1.4), we formulate now the complete set of assumptions used in this investigation.

ASSUMPTION A. $\mathcal{X}$ is a convex set in $\mathbf{X}$; for each $i \in I'$ the continuous functions c_i and $-q_{is}$ are convex on $\mathcal{X}$ for $s \in S_{i1}$ and q_{is} is affine in $\mathbf{X}$ for $s \in S_{i2}$.

ASSUMPTION B. For each $i \in I'$ there exist numbers $\theta_i > 0,\ \zeta_i > 0$ and a vector $\tilde{z}_i \in \mathbf{Z}_i$ such that

(i) for every $y_i^{(2)} = \{y_{is}, s \in S_{i2}\} \in \mathbb{R}^{|S_{i2}|}$ the linear system

$$\langle p_{is}, \xi_i \rangle = y_{is} \ \ \forall s \in S_{i2}$$

has a solution $\xi_i = \xi_i(y_i^{(2)})$ with $\|\xi_i\| \leqslant \theta_i max_{s \in S_{i2}} |y_{is}|$ (it is obvious that this condition is equivalent to the linear independence of the vectors $p_{is}, s \in S_{i2}$);

(ii) $\quad \langle p_{is}, \tilde{z}_i \rangle \leqslant \zeta_i < 0 \quad \forall s \in S_{i1}, \qquad \langle p_{is}, \tilde{z}_i \rangle = 0 \quad \forall s \in S_{i2}.$

From the convex analysis it is well known (see, for instance, Rockafellar, 1970) that under Assumption B the set $D_i(x)$ is nonempty for arbitrary $i \in I'$ and $x \in \mathcal{X}$.

PROPOSITION 2.1. *Suppose that Assumption B is fulfilled, then for each* $i \in I'$ *it holds*

$$\left\| \sum_{s \in S_i} y_{is} p_{is} \right\| \geqslant \kappa_i \max_{s \in S_i} |y_{is}| \quad \forall y_i \in \mathcal{Y}_i, \tag{2.8}$$

with $\kappa = \zeta_i \left[\zeta_i \theta_i + \|\tilde{z}_i\| (1 + \theta_i max_{s \in S_{i1}} \|p_{is}\|) \right]^{-1}$.

Proof. Let $y_i = \{y_i^{(1)}, y_i^{(2)}\}$, with $y_i^{(1)} = \{y_{is}, s \in S_{i1}\} \geqslant 0,\ y_i^{(2)} = \{y_{is}, s \in S_{i2}\}$ and $max_{s \in S_i} |y_{is}| = 1$. Due to Assumption B(i) there exists a vector $\xi_i = \xi_i(y_i^{(2)})$ such that

$$\langle p_{is}, \xi_i \rangle = sign \ y_{is} \quad \forall s \in S_{i2}, \qquad \|\xi_i\| \leqslant \theta_i max_{s \in S_{i2}} |y_{is}| \leqslant \theta_i.$$

Denote

$$\gamma_i = \theta_i \left[1 + \theta_i max_{s \in S_{i1}} \|p_{is}\|\right]^{-1},$$
$$\bar{z}_i(y_i) = (\gamma_i \xi_i(y_i^{(2)}) - \tilde{z}_i)\|\gamma_i \xi_i(y_i^{(2)}) - \tilde{z}_i\|^{-1}.$$

Due to

$$\|\bar{z}_i(y_i)\| = 1, \qquad \gamma_i = \zeta_i - \gamma_i \theta_i max_{s \in S_{i1}} \|p_{is}\|,$$
$$\text{and} \quad \sum_{s \in S_i} |y_{is}| \geqslant max_{s \in S_i} |y_{is}| = 1,$$

we obtain

$$\begin{aligned}
\|\sum_{s \in S_i} y_{is} p_{is}\| &\geqslant \left\langle \sum_{s \in S_i} y_{is} p_{is}, \bar{z}_i(y_i) \right\rangle \\
&= \|\gamma_i \xi_i(y_i^{(2)}) - \tilde{z}_i\|^{-1} \left[\sum_{s \in S_{i1}} y_{is} \left\langle p_{is}, \gamma_i \xi_i(y_i^{(2)}) \right\rangle \right. \\
&\quad \left. + \sum_{s \in S_{i1}} y_{is} \langle p_{is}, -\tilde{z}_i \rangle + \sum_{s \in S_{i2}} y_{is} \gamma_i sign\ y_{is} \right] \\
&\geqslant (\gamma_i \theta_i + \|\tilde{z}_i\|)^{-1} \left[(\zeta_i - \gamma_i \theta_i max_{s \in S_{i1}} \|p_{is}\|) \sum_{s \in S_{i1}} y_{is} + \gamma_i \sum_{s \in S_{i2}} |y_{is}| \right] \\
&= (\gamma_i \theta_i + \|\tilde{z}_i\|)^{-1} \gamma_i \sum_{s \in S_i} |y_{is}| \geqslant (\gamma_i \theta_i + \|\tilde{z}_i\|)^{-1} \gamma_i.
\end{aligned}$$

Hence, if $max_{s \in S_i} |y_{is}| = 1$, then $\| \sum_{s \in S_i} y_{is} p_{is}\| \geqslant (\gamma_i \theta_i + \|\tilde{z}_i\|)^{-1} \equiv \kappa_i$. □

ASSUMPTION C. There exists a vector $\tilde{x} \in \mathcal{X}$ such that

$$m_0(\tilde{x}) < +\infty, \quad c_i(\tilde{x}) + m_i(\tilde{x}) \leqslant 0 \ \forall i \in I$$

and the set

$$Q_{\tilde{x}} := \{x \in Q : c_0(x) + m_0(x) \leqslant c_0(\tilde{x}) + m_0(\tilde{x})\}$$

is bounded.

Condition C guarantees that the set $\{x \in Q : c_0(x) + m_0(x) < +\infty\}$ is non-empty, that the optimal value of Problem (1.1)–(1.4) is finite, and that the optimal solution of Problem (1.1)–(1.4) belongs to the set $Q_{\tilde{x}}$. It is obvious that

$$m_i(x) < +\infty \ \ \forall i \in I', \forall x \in Q_{\tilde{x}}.$$

From the theory of convex quadratic programming (cf., for instance, Künzi and Krelle, 1962; Pshenichny, 1980) it holds: If in Problem (1.5) $D_i(x) \neq \emptyset$ and the optimal value is finite, then the strong dualtiy relation is valid, i.e.,

$$m_i(x) := \sup_{z_i \in D_i(x)} \left\{ \tfrac{1}{2}\langle G_i z_i, z_i\rangle + \langle \ell_i + H_i x, z_i\rangle \right\} = \inf_{y_i \in \mathcal{Y}_i} \varphi_i(x, y_i). \tag{2.9}$$

Moreover, the optimal sets $\mathcal{Z}_i^*(x)$ and $\mathcal{Y}_i^*(x)$ of the Problems (1.5) and (2.6) are nonempty, respectively. Therefore, from the Assumptions B and C it follows that for each $i \in I'$ and $x \in Q_{\tilde{x}}$ the duality relation (2.9) holds, and that $\mathcal{Z}_i^*(x) \neq \emptyset$ and $\mathbf{Y}_i^*(x) \neq \emptyset$.

ASSUMPTION D. For each $i \in I'$ there are known numbers $\underline{u}_i$, $M_i > 0$, $\underline{v}_{is}$, and $\bar{v}_{is}$ $(s \in S_i)$ such that for arbitrary $x \in \mathcal{X}$ the inequalities

$$\begin{aligned} c_i(x) &\geqslant \underline{u}_i, \\ \|\ell_i + H_i x\| &\leqslant M_i, \\ \underline{v}_{is} \leqslant -q_{is}(x) &\leqslant \bar{v}_{is} \ (s \in S_i) \end{aligned} \tag{2.10}$$

are true.

Assumption D can be replaced by a weaker, but more complicated verificable one, namely that the inequalities (2.10) are satisfied only for each $x \in Q_{\tilde{x}}$.

Concerning the determination of the constants arising in (2.10) we emphasize:

- Condition $c_i(x) \geqslant \underline{u}_i \ \forall x \in \mathcal{X}$ is always true, if the set $\mathcal{X}$ is bounded or $c_i(x) \to +\infty$ for $x \in \mathcal{X}, \|x\| \to \infty$; to calculate the numbers $\underline{u}_i$ it is sufficient to find or to underestimate the values $inf\{c_i(x) : x \in \mathcal{X}\}$.
- Condition $\|\ell_i + H_i x\| \leqslant M_i \ \forall x \in \mathcal{X}$ holds, if the set $\mathcal{X}$ is bounded or $\|H_i x\| \geqslant a_i\|x\| \ \forall x \in \mathbf{X} \ (a_i > 0)$, i.e., $\{x : Hx = 0\} = \{0_{\mathbf{X}}\}$; to determine the numbers M_i it is sufficient to find or to overestimate $sup\{\|\ell_i + H_i x\| : x \in \mathcal{X}\}$. This supremum can be easily calculated if the set $\mathcal{X}$ is bounded, namely

$$sup\{\|\ell_i + H_i x\| : x \in \mathcal{X}\} \leqslant \|\ell_i\| + \|H_i\| sup_{x \in \mathcal{X}}\|x\|.$$

  Note that, in general, the problem $sup\{\|\ell_i + H_i x\| : x \in \mathcal{X}\}$ consists of the maximization of a convex function on a convex set. This is one of the well studied problems in global optimization. It is well-known, if $\mathcal{X}$ is a convex polyhedron, that this maximum is equal to the maximum of the values of the objective function in the vertices of $\mathcal{X}$.
- Finally, it should be noted that for $s \in S_{i1}$, the numbers $\underline{v}_{is}$ are minorants of the optimal values of the following convex optimization problems

$$-q_{is}(x) \to \min, \quad x \in \mathcal{X}, \tag{2.11}$$

and $\bar{v}_{is}$ are majorants of the optimal values of the problems

$$-q_{is}(x) \to \max, \quad x \in \mathcal{X}. \tag{2.12}$$

In the latter problem a convex function on a convex set has to be maximized. For the solution of this problem we refer to several numerical methods in Horst and Tuy (1996).

Analogously, for $s \in S_{i2}$, the numbers $\underline{v}_{is}$ and $\bar{v}_{is}$ are minorants and majorants, respectively, of the optimal values of the convex Problems (2.11) and (2.12) (observe that for $s \in S_{i2}$ the functions q_{is} are supposed to be affine). If $\mathcal{X}$ is a convex polyhedron, then obviously we deal in (2.12) with linear programming problems.

3. Uniform embedding of the optimal sets $\mathcal{Y}_i^*(x)$ into polyhedrons

For given $N_i > 0$ $(i \in I')$ we consider the following polyhedrons

$$P_i(N_i) := \{y_i \in \mathbf{Y}_i : 0 \leqslant y_{is} \leqslant N_i \ (s \in S_{i1}),\ |y_{is}| \leqslant N_i \ (s \in S_{i2})\}. \tag{3.1}$$

In the sequel we describe a procedure determining the constants N_i in (3.1) such that

$$\mathcal{Y}_i^*(x) \subset P_i(N_i) \quad \forall i \in I', \quad \forall x \in Q_{\bar{x}}.$$

Let $\mathfrak{R}_i = \{z_i \in \mathbf{Z}_i : G_i z_i = 0\}$ be the kernel of the operator G_i, $\mathfrak{R}_i^\perp$ be the orthogonal complement to the subspace $\mathfrak{R}_i$, then

$$\mathfrak{R}_i^\perp = \left\{z_i' \in \mathbf{Z}_i : z_i' = G_i^T z_i \text{ for some } z_i \in \mathbf{Z}_i\right\}, \tag{3.2}$$

with G_i^T the conjugate operator to G_i. Denote $\hat{G}_i$ the restriction of the operator G_i on the subspace $\mathfrak{R}_i^\perp$. Then we obtain:

$$\text{if } z_i \in \mathbf{Z}_i,\ z_i = \check{z}_i + \hat{z}_i, \quad \text{with } \check{z}_i \in \mathfrak{R}_i,\ \hat{z}_i \in \mathfrak{R}_i^\perp \Rightarrow \hat{G}_i \hat{z}_i = G_i z_i.$$

In this way we deal with a non-degenerate operator $\hat{G}_i$, i.e. there exists the inverse $(\hat{G}_i)^{-1}$. Taking into account that G_i is assumed to be negative semi-definite, the operators $\hat{G}_i$ and $(\hat{G}_i)^{-1}$ are negative definite, i.e., for some $\lambda_i > 0$ it holds

$$\langle \hat{G}_i \hat{z}_i, \hat{z}_i \rangle \leqslant -\lambda_i \|\hat{z}_i\|^2 \quad \forall \hat{z}_i \in \mathfrak{R}_i^\perp. \tag{3.3}$$

Now, we come back to our problem. For each $i \in I'$, $x \in \mathbf{X}$, $y_i \in \mathbf{Y}_i$ denote

$$\bar{z}_i(x, y_i) = \ell_i + H_i x - \sum_{s \in S_i} y_{is} p_{is}. \tag{3.4}$$

Obviously, $\bar{z}_i$ is an affine operator from $\mathbf{X} \times \mathbf{Y}_i$ into $\mathbf{Z}_i$.

PROPOSITION 3.1. *For each* $i \in I', x \in \mathcal{X}, y_i \in \mathcal{Y}_i$ *the function* $\pi_i(x, y_i)$, *defined by (2.5), is equal to*

$$\pi_i(x, y_i) = \begin{cases} -\frac{1}{2}\left\langle \left(\hat{G}_i\right)^{-1} \bar{z}_i(x, y_i), \bar{z}_i(x, y_i) \right\rangle & \text{if } \bar{z}_i(x, y_i) \in \Re_i^{\perp}, \\ +\infty \text{ otherwise}. \end{cases}$$

Proof. Let $\bar{z}_i(x, y_i) \notin \Re_i^{\perp}$, i.e. $\exists z_i \in \Re_i, z_i \neq 0$ such that $\langle \bar{z}_i(x, y_i), z_i \rangle \neq 0$. Without loss of generality let $\langle \bar{z}_i(x, y_i), z_i \rangle > 0$, otherwise $-z_i \in \Re_i$ can be chosen. Due to $\langle G_i z_i, z_i \rangle = 0$ the function

$$\tfrac{1}{2}\langle G_i(tz_i), tz_i \rangle + \langle \bar{z}_i(x, y_i), tz_i \rangle = t\langle \bar{z}_i(x, y_i), z_i \rangle \to +\infty \quad \text{for } t \to +\infty.$$

Hence, $\pi_i(x, y_i) = +\infty$.

Now, let $\bar{z}_i(x, y_i) \in \Re_i^{\perp}$. For arbitrary $z_i \in \mathbf{Z}_i$ denote $\hat{z}_i$ the orthogonal projection of z_i on the subspace $\Re_i^{\perp}$. Then we get

$$\langle G_i z_i, z_i \rangle = \langle \hat{G}_i \hat{z}_i, \hat{z}_i \rangle \quad \text{and} \quad \langle \bar{z}_i(x, y_i), z_i \rangle = \langle \bar{z}_i(x, y_i), \hat{z}_i \rangle$$
$$\text{because of } z_i - \hat{z}_i \in \Re_i.$$

Hence, if $\bar{z}_i(x, y_i) \in \Re_i^{\perp}$, one can conclude that

$$\sup_{z_i \in \mathbf{Z}_i} \left\{ \tfrac{1}{2}\langle G_i z_i, z_i \rangle + \langle \bar{z}_i(x, y_i), z_i \rangle \right\} = \sup_{\hat{z}_i \in \Re_i^{\perp}} \left\{ \tfrac{1}{2}\langle \hat{G}_i \hat{z}_i, \hat{z}_i \rangle + \langle \bar{z}_i(x, y_i), \hat{z}_i \rangle \right\}$$
$$= -\tfrac{1}{2}\left\langle \left(\hat{G}_i\right)^{-1} \bar{z}_i(x, y_i), \bar{z}_i(x, y_i) \right\rangle,$$

because the function $\frac{1}{2}\langle \hat{G}_i \hat{z}_i, \hat{z}_i \rangle + \langle \bar{z}(x, y_i), \hat{z}_i \rangle$ attains its maximum w.r.t. $\hat{z}_i \in \Re_i^{\perp}$ in the point

$$\hat{z}_i = -\left(\hat{G}_i\right)^{-1} \bar{z}_i(x, y_i). \tag{3.5}$$

□

COROLLARY 3.1. *Assume that the matrix* G_i *is negative definite. Then*

$$\pi_i(x, y_i) = -\tfrac{1}{2}\langle G_i^{-1} \bar{z}_i(x, y_i), \bar{z}_i(x, y_i) \rangle.$$

Indeed, in this case one has $\Re_i = \{0\}$, $\Re_i^{\perp} = \mathbf{Z}_i$, and $\hat{G}_i = G_i$.

Denote

$$\mathcal{Y}_i(x) = \{y_i \in \mathcal{Y} : \bar{z}_i(x, y_i) = G_i^T z_i,\ z_i \in Z_i\}.$$

Proposition 3.1 leads immediately to

PROPOSITION 3.2. *For each $i \in I'$, $x \in \mathcal{X}$ it holds*

$$\mathcal{Y}_i^*(x) = \operatorname{Argmin}\left\{ -\frac{1}{2}\left\langle \left(\hat{G}_i\right)^{-1} \bar{z}_i(x, y_i), \bar{z}_i(x, y_i) \right\rangle - \sum_{s \in S_i} y_{is} q_{is}(x) : y_i \in \mathcal{Y}_i(x) \right\}.$$

Denote γ_i the minimal eigenvalue of the operator $-(\hat{G}_i)^{-1}$. Due to (3.3) we have $\gamma_i > 0$.

PROPOSITION 3.3. *If, for each $i \in I'$, $x \in \mathcal{X}$, $y_i \in \mathcal{Y}_i$,*

$$\left(\ell_i + H_i x - \sum_{s \in S_i} y_{is} p_{is} \right) \in \mathfrak{R}_i^{\perp}$$

then

$$\begin{aligned}
&\varphi_i(x, y_i) \\
&\quad = -\frac{1}{2}\left\langle \left(\hat{G}_i\right)^{-1} \left[\ell_i + H_i x - \sum_{s \in S_i} y_{is} p_{is} \right], \ell_i + H_i x - \sum_{s \in S_i} y_{is} p_{is} \right\rangle \\
&\qquad - \sum_{s \in S_i} y_{is} q_{is}(x) \\
&\quad \geqslant \xi_i \max_{s \in S_i} \left\{ y_{is}^2 \right\} - \eta_i \max_{s \in S_i} |y_{is}|,
\end{aligned}$$

with

$$\xi_i = \frac{1}{2}\gamma_i \kappa_i^2, \eta_i = \gamma_i M_i \kappa_i + \sum_{s \in S_i} \max\left\{ |\underline{v}_{is}|, |\bar{v}_{is}| \right\} \text{ and } \kappa_i \text{ from (2.8).}$$

Proof. From the inequality

$$\langle -(\hat{G}_i)^{-1} \hat{z}_i, \hat{z}_i \rangle \geqslant \gamma_i \|\hat{z}_i\|^2 \quad \forall \hat{z}_i \in \mathfrak{R}_i^{\perp}$$

it follows that

$$\begin{aligned}
&\frac{1}{2}\left\langle -(\hat{G}_i)^{-1} \left[\ell_i + H_i x - \sum_{s \in S_i} y_{is} p_{is} \right], \ell_i + H_i x - \sum_{s \in S_i} y_{is} p_{is} \right\rangle \\
&\quad \geqslant \frac{1}{2}\gamma_i \|\ell_i + H_i x - \sum_{s \in S_i} y_{is} p_{is}\|^2 \\
&\quad \geqslant \frac{1}{2}\gamma_i \left[-2\|\ell_i + H_i x\| \, \|\sum_{s \in S_i} y_{is} p_{is}\| + \|\sum_{s \in S_i} y_{is} p_{is}\|^2 \right].
\end{aligned}$$

In view of Assumption D, and (2.4), (2.5) and (2.8), we obtain that

$$\begin{aligned}\varphi_i(x,y) =& \pi_i(x,y_i) - \sum_{s\in S_i} y_{is}q_{is}(x)\\ &\geqslant \left(\tfrac{1}{2}\gamma_i\kappa_i^2\right)\max_{s\in S_i}\left\{y_{is}^2\right\} - \gamma_i M_i\kappa_i \max_{s\in S_i}\{|y_{is}|\}\\ &- \left(\sum_{s\in S_i}\max\left\{|\underline{v}_{is}|,|\bar{v}_{is}|\right\}\right)\max_{s\in S_i}\{|y_{is}|\}\\ &= \xi_i \max_{s\in S_i}\left\{y_{is}^2\right\} - \eta_i \max_{s\in S_i}\{|y_{is}|\}.\end{aligned}$$

□

THEOREM 3.1. *The following statements are true:*

(i) If $x\in Q_{\tilde{x}}$ (with $\tilde{x}$ from Assumption C), then

$$\mathcal{Y}_0^*(x)\subset P_0(N_0),$$

with

$$N_0 = (2\xi_0)^{-1}\left(\eta_0 + \left[\eta_0^2 + 4\xi_0\max\left\{c_0(\tilde{x}) + m_0(\tilde{x}) - \underline{u}_0, 0\right\}\right]^{1/2}\right);$$

(ii) if $x\in Q$, then for each $i\in I$ it holds

$$\mathcal{Y}_i^*(x)\subset P_i(N_i),$$

with

$$N_i = (2\xi_i)^{-1}\left(\eta_i + \left[\eta_i^2 + 4\xi_i\max\left\{-\underline{u}_i, 0\right\}\right]^{1/2}\right).$$

Proof. (i) If $x\in Q_{\tilde{x}}$ and $y_0\in\mathcal{Y}_0^*(x)$, then from Assumption D and Proposition 3.3 we obtain

$$\begin{aligned}&c_0(\tilde{x}) + m_0(\tilde{x}) \geqslant c_0(x) + m_0(x) = c_0(x) + \varphi_0(x,y_0)\\ &\quad\geqslant \underline{u}_0 + \xi_0\max_{s\in S_0}\left\{y_{0s}^2\right\} - \eta_0\max_{s\in S_0}\{|y_{0s}|\},\end{aligned}$$

i.e.

$$\xi_0\max_{s\in S_0}\left\{y_{0s}^2\right\} - \eta_0\max_{s\in S_0}\{|y_{0s}|\} \leqslant c_0(\tilde{x}) + m_0(\tilde{x}) - \underline{u}_0.$$

Hence,

$$\begin{aligned}\max_{s\in S_0}\{|y_{0s}|\} &= \max\left\{\max_{s\in S_{01}}\{y_{0s}\}, \max_{s\in S_{02}}\{|y_{0s}|\}\right\}\\ &\leqslant N_0 = (2\xi_0)^{-1}\left(\eta_0 + \left[\eta_0^2 + 4\xi_0\max\left\{c_0(\tilde{x}) + m_0(\tilde{x}) - \underline{u}_0, 0\right\}\right]^{1/2}\right).\end{aligned}$$

(ii) If $x \in Q$, then from Assumption D and Proposition 3.3, one can conclude for each $i \in I$, $y_i \in \mathcal{Y}_i^*(x)$ that

$$0 \geqslant c_i(x) + m_i(x) = c_i(x) + \varphi_i(x, y_i) \geqslant \underline{u}_i + \xi_i \max_{s\in S_i}\left\{y_{is}^2\right\} - \eta_i \max_{s\in S_i}\{|y_{is}|\},$$

and this leads to

$$\max_{s\in S_i}\{|y_{is}|\} = \max\left\{\max_{s\in S_{i1}}\{y_{is}\}, \max_{s\in S_{i2}}\{|y_{is}|\}\right\}$$
$$\leqslant N_i = (2\xi_i)^{-1}\left(\eta_i + \left[\eta_i^2 + 4\xi_i \max\left\{-\underline{u}_i, 0\right\}\right]^{1/2}\right). \qquad \square$$

4. The reduced problem and a branch-and-bound approach

Denote $\mathbf{V}_i = \mathbb{R}^{|S_i|}$ the Euclidean space of the vectors $v_i = \{v_{is}\}_{s\in S_i}$ $(i \in I')$ and let $\mathbf{Y}$, $\mathbf{V}$ be the product (w.r.t. $i \in I'$) spaces of $\mathbf{Y}_i$ and $\mathbf{V}_i$, respectively.

THEOREM 4.1. *Let the Assumptions A–D be fulfilled. Then Problem (1.1)–(1.4) is equivalent to the following reduced problem in the space* $\mathbf{X} \times \mathbf{Y} \times \mathbf{Z} \times \mathbf{V}$*:*

$$c_0(x) - \tfrac{1}{2}\left\langle \left(\hat{G}_0\right)^{-1}\left[\ell_0 + H_0 x - \sum_{s\in S_0} y_{0s} p_{0s}\right], \ell_0 + H_0 x - \sum_{s\in S_0} y_{0s} p_{0s}\right\rangle$$
$$+ \langle y_0, v_0\rangle \to \min \tag{4.1}$$

s.t.

$$x \in \mathcal{X}, \quad y_i \in P_i(N_i), \quad z_i \in \mathbf{Z}_i, \quad v_i \in \left[\underline{v}_i, \bar{v}_i\right] \quad (i \in I'), \tag{4.2}$$

$$\ell_i + H_i x - \sum_{s\in S_i} y_{is} p_{is} - G_i^T z_i = 0 \quad (i \in I'), \tag{4.3}$$

$$c_i(x) - \tfrac{1}{2}\left\langle \left(\hat{G}_i\right)^{-1}\left[\ell_i + H_i x - \sum_{s\in S_i} y_{is} p_{is}\right], \ell_i + H_i x - \sum_{s\in S_i} y_{is} p_{is}\right\rangle$$
$$+ \langle y_i, v_i\rangle \leqslant 0 \quad (i \in I), \tag{4.4}$$

$$-q_{is}(x) \leqslant v_{is} \ (i \in I', s \in S_{i1}), \quad -q_{is}(x) = v_{is} \quad (i \in I', s \in S_{i2}), \tag{4.5}$$

with the constants N_i*, given in Theorem 3.1, and* $\underline{v}_i = \left\{\underline{v}_{is}\right\}_{s\in S_i}$, $\bar{v}_i = \{\bar{v}_{is}\}_{s\in S_i}$ *are the minorants and majorants for the optimal values of the Problems (2.11), (2.12), respectively.*

Proof. Under the assumptions made we have for each $i \in I', x \in \mathcal{X}$ that

$$m_i(x) = \min \left\{ \varphi_i(x, y_i) : y_i \in \mathcal{Y}_i \right\} = \min \left\{ \pi_i(x, y_i) - \sum_{s \in S_i} y_{is} q_{is}(x) : y_i \in \mathcal{Y} \right\}.$$

Therefore, Problem (1.1)–(1.4) is equivalent to the minimization problem in the space $\mathbf{X} \times \mathbf{Y}$:

$$c_0(x) + \pi_0(x, y_0) - \sum_{s \in S_0} y_{0s} q_{0s}(x) \to \min \tag{4.6}$$

s.t.

$$x \in \mathcal{X}, \quad y_i \in \mathcal{Y}_i \ (i \in I'), \quad c_i(x) + \pi_i(x, y_i) - \sum_{s \in S_i} y_{is} q_{is}(x) \leqslant 0 \ (i \in I). \tag{4.7}$$

Taking into account that, for each $i \in I', x \in \mathcal{X}$, the function $\pi_i(x, y_i) - \sum_{s \in S_i} y_{is} q_{is}(x)$ attains its mimimum w.r.t. $y_i \in \mathcal{Y}_i$ on the set $\mathcal{Y}_i^*(x)$, from Theorem 3.1 it follows that we do not lose any point of the optimal set of Problem (4.6)–(4.7), due to the inclusion $\mathcal{Y}_i^* \subset P_i(N_i)$.

Therefore, using (3.3) and the Propositions 3.1, 3.2, Problem (4.6), (4.7) is equivalent to the following problem in $\mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$:

$$c_0(x) - \tfrac{1}{2} \left\langle \left(\hat{G}_0 \right)^{-1} \left[\ell_0 + H_0 x - \sum_{s \in S_0} y_{0s} p_{0s} \right], \ell_0 + H_0 x - \sum_{s \in S_0} y_{0s} p_{0s} \right\rangle + \sum_{s \in S_0} y_{0s} \left[-q_{0s}(x) \right] \to \min$$

s.t.

$$x \in \mathcal{X}, \quad y_i \in P_i(N_i), \quad z_i \in \mathbf{Z}_i \quad (i \in I'),$$

$$\ell_i + H_i x - \sum_{s \in S_i} y_{is} p_{is} - G_i^T z_i = 0 \quad (i \in I'),$$

$$c_i(x) - \tfrac{1}{2} \left\langle \left(\hat{G}_i \right)^{-1} \left[\ell_i + H_i x - \sum_{s \in S_i} y_{is} p_{is} \right], \ell_i + H_i x - \sum_{s \in S_i} y_{is} p_{is} \right\rangle + \sum_{s \in S_i} y_{is} \left[-q_{is}(x) \right] \leqslant 0 \quad (i \in I).$$

Inserting in this problem the additional variables $v_i \in \left[\underline{v}_i, , \bar{v}_i\right]$ $(i \in I')$ and the additional constraints (4.5), then we obtain Problem (4.1)–(4.5). □

Let μ^* be the optimal value of the reduced Problem (4.1)–(4.5). Due to Theorem (4.1), this value is equal to the optimal value of Problem (1.1)–(1.4). Moreover, if x^*, y_i^*, z_i^*, v_i^* $(i \in I')$ are the components of the optimal solution of Problem (4.1)–(4.5), then x^* is the optimal solution of Problem (1.1)–(1.4).

Suppose that, for fixed $\varepsilon \geqslant 0$, the vector $\{x, y, z, v\} \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z} \times \mathbf{V}$ satisfies the constraints (4.2), (4.3), (4.5). Then this vector is called an *approximate global minimum* (of exactness ε) of Problem (4.1)–(4.5), if

$$c_i(x) + \pi_i(x, y_i) + \sum_{s \in S_i} y_{is} v_{is} \leqslant \varepsilon \ (i \in I)$$

and

$$c_0(x) + \pi_0(x, y_0) + \sum_{s \in S_0} y_{0s} v_{0s} \leqslant \mu^* + \varepsilon.$$

Now, we describe how an approximate global solution of Problem (4.1)–(4.5) can be found by means of a branch-and-bound technique.

Let $\mathcal{G} \subset \mathbf{X} \times \mathbf{Y} \times \mathbf{Z} \times \mathbf{V}$ be the feasible set of Problem (4.1)–(4.5). Observe that the constraints (4.2), (4.3) and (4.5) are convex. In the objective function only the last term $\langle y_0, v_0 \rangle$ is nonconvex on $P_0(N_0) \times \left[\underline{v}_0, \bar{v}_0\right]$. Analogously, in the constraints (4.4) only the last term $\langle y_i, v_i \rangle$ is nonconvex on the set $P_i(N_i) \times \left[\underline{v}_i, \bar{v}_i\right]$. However, it is possible to construct convex hulls for the functions $\langle y_i, v_i \rangle$ on the parallelepipeds $P_i(N_i) \times \left[\underline{v}_i \bar{v}_i\right]$ for each $i \in I'$. This permits us to find on the set $\mathcal{G}$ convex minorants for the functions $c_i(x) + \pi_i(x, y_i) + \langle y_i, v_i \rangle$ $(i \in I')$.

Recall that a *convex hull of a nonconvex function f on the convex set* $\mathcal{U}$ in the Euclidean space $\mathbf{U}$ is the largest convex function $\underline{f}$ of all convex functions on $\mathcal{U}$ for which $f(u) \geqslant \underline{f}(u)$ $\forall u \in \mathcal{U}$ (cf. Rockafellar, 1970; Horst and Tuy, 1996). We denote this convex hull by $co_{\mathcal{U}} f$.

From this definition it follows immediately that, if $\mathbf{U}$ is the product (w.r.t. $s \in S$ (S-finite)) of the Euclidean spaces $\mathbf{U}_s$ and the set $\mathcal{U}$ is the product of the convex sets $\mathcal{U}_s$ and

$$f(u) = \sum_{s \in S} f_s(u_s) \quad \forall u = \{u_s\}_{s \in S} \in \mathcal{U},$$

then

$$co_{\mathcal{U}} f(u) = \sum_{s \in S} co_{\mathcal{U}_s} f_s(u_s) \quad \forall u = \{u_s\}_{s \in S} \in \mathcal{U}. \tag{4.8}$$

Now, let us return to our problem. For each $i \in I'$ we denote by $\mathcal{P}_i$ the parallelepiped $P_i(N_i) \times \left[\underline{v}_i, \bar{v}_i\right]$ in the space $\mathbf{Y}_i \times \mathbf{V}_i$. Then $\mathcal{P}_i$ is the product w.r.t.

$s \in S_i$ of the rectangles

$$\mathcal{P}_{is} = \left\{ \{y_{is}, v_{is}\} \in \mathbb{R}^2 : \underline{y}_{is} \leqslant y_{is} \leqslant N_i,\ \underline{v}_{is} \leqslant v_{i,s} \leqslant \bar{v}_{is} \right\} \subset \mathbb{R}^2,$$

where $\underline{y}_{is} = 0$ for $s \in S_{i1}$ and $\underline{y}_{is} = -N_i$ for $s \in S_{i2}$.

It is well known that the convex hull of a function of two variables $f(t_1, t_2) := t_1 t_2$ on an arbitrary rectangle in $\mathbb{R}^2$ is a piecewise linear function consisting of two linear parts (see, for instance, Horst and Tuy, 1996). The corresponding two planes can be easily constructed by means of the values of the function $f = t_1 t_2$ in the vertices of the rectangle. Indeed, if $u^k = \left\{t_1^k, t_2^k\right\}$ $(k = 1, 2, 3, 4)$ are the vertices of the rectangle and

$$f(u^1) \leqslant f(u^2) \leqslant f(u^3) \leqslant f(u^4), \quad \text{with } f(u^1) < f(u^4),$$

then the first plane in $\mathbb{R}^3$ crosses the points $\left\{u^1, f(u^1)\right\}, \left\{u^2, f(u^2)\right\}, \left\{u^3, f(u^3)\right\}$, and the second one is defined by the points $\left\{u^2, f(u^2)\right\}, \left\{u^3, f(u^3)\right\}, \left\{u^4, f(u^4)\right\}$. Due to

$$\langle y_i, v_i \rangle = \sum_{s \in S_i} y_{is} v_{is}, \qquad \mathcal{P}_i = \prod_{s \in S_i} \mathcal{P}_{is},$$

and (4.8), we obtain

$$co_{\mathcal{P}_i} \langle y_i, v_i \rangle = \sum_{s \in S_i} co_{\mathcal{P}_{is}} (y_{is} v_{is}).$$

In view of

$$\langle y_i, v_i \rangle \geqslant co_{\mathcal{P}_i} \langle y_i, v_i \rangle \quad \forall \{y_i, v_i\} \in \mathcal{P}_i\ (i \in I'),$$

the function

$$c_i(x) + \pi_i(x, y_i) + co_{\mathcal{P}_i} \langle y_i, v_i \rangle$$

is a convex minorant for $c_i(x) + \pi_i(x, y_i) + \langle y_i, v_i \rangle$ on the convex set $\mathcal{X} \times P_i(N_i) \times \left[\underline{v}_i, \bar{v}_i\right]$.

The simplicity of the construction of such convex minorants permits us to use the branch-and-bound method (cf. Horst and Tuy, 1996; Levitin and Khranovich, 1996) for solving Problem (4.1)–(4.5). This method reduces the problem of finding an approximate global solution of Problem (4.1)–(4.5) to the solution of a finite number of convex estimating problems. In this way, in the kth step, one has to solve

$$c_0(x) + \pi_0(x, y_0) + \sum_{s \in S_0} co_{\mathcal{P}_{0s}(k)} (y_{0s} v_{0s}) \to \min$$

s.t.

$$x \in \mathcal{G}, z_i \in \mathbf{Z}_i\ (i \in I'),\ \{y_{is}, v_{is}\} \in \mathcal{P}_{is}(k)\ (i \in I', s \in S_i), \tag{4.9}$$

$$v_i \in \left[\underline{v}_i, \bar{v}_i\right] (i \in I'), \quad \text{constraints (4.3), (4.5), and}$$

$$c_i(x) + \pi_i(x, y_i) + \sum_{s \in S_i} co_{\mathcal{P}_{is}(k)}(y_{is} v_{is}) \leqslant 0 \ (i \in I).$$

Here, $\mathcal{P}_{is}(k)$ is some rectangle, obtained in the k-th step by partition of the rectangle $\mathcal{P}_{is}$. Of course, the efficiency of this approach depends on the number of partitions which must be made in order to get a suitable accuracy for the approximation of the global minimum. As far as we know, there are no results in the literature for the speed of convergence for branch-and-bound methods. For a more detailed description of the branch-and-bound method dealing with a wider class of problems, containing also problems of the form (4.1)–(4.5), we refer to Levitin and Khranovich (1996). Under some additional assumptions, there is also shown a result about the finiteness of the number of steps for finding an approximate global minimum.

Concerning the consideration of relaxed problems, i.e. problems where the quadratic functions in the lower level problems can be replaced by convex (resp. concave) functions, we refer to Levitin (1997). But it should be mentioned that in this case the computation of the bounds for constructing the parallelepipeds in the embedding procedure is much more complicated, because, as a rule, there does not exist an explicit formula for the function $\pi_i(x, y_i)$. Therefore, the main purpose of the paper is to examine the proof of the possibility to reduce the solution of the special generalized semi-infinite programming problem (1.6) to the solution of a finite number of convex programming problems (4.9). Note that, if we suppose that $\mathcal{X}$ is a convex polyhedral set, that the functions G_i in (1.3) are equal to zero for all $i \in I$, and the functions $c_i(x), -q_{is}(x)$ $(i \in I', s \in S_{i1})$ are supposed to be affine, then, all the estimating problems (4.9) are quadratic programming problems. If, moreover, it is supposed that $G_0 \equiv 0$, then all the problems (4.9) are linear programming problems.

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